Heterogeneous agent model of multi-assets in discrete-time

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Outline

He, X.Z., Li, K., Wang, C. CHAOS 28 (2018)
Time-varying economic dominance in financial markets: A bistable dynamics approach

$\Rightarrow$ Continuous-time heterogeneous agent financial market of multi-assets traded by fundamental and momentum investors

- We discretize the benchmark model: this offers new opportunities of investigation because of different tools of analysis
- Our results contribute to the literature on heterogeneous agent models of multi-assets, since it mainly focuses on stylized models
- In fact, the analysis in discrete time is often limited to an estimation of the resulting system

Financial market of He et al. (2018) with two risky assets A,B

$$\forall i,j = A,B \quad (i \neq j) \Longrightarrow \dot{P}^i(t) = \frac{dP^i}{dt} =$$

$$= \mu^i \left[\alpha_f^i \tanh\left[\beta_f\left(\bar{F}^i - P^i(t)\right)\right] + \alpha_a^i \tanh\left[\beta_a\left(P^i(t) - P^i(t-\tau)\right)\right] + \right.$$
$$\left. + \alpha_c^i \tanh\left\{\beta_c\left[\left(P^i(t) - P^i(t-\tau)\right) - \left(P^j(t) - P^j(t-\tau)\right)\right]\right\}\right]$$

where *excess demands* of investors are:

- **Fundamentalists** $D^i_{f,t} = \tanh\left[\beta_f(F^i_t - P^i_t)\right]$
- **Absolute momentum** $D^i_{a,t} = \tanh\left[\beta_a(P^i_t - P^i_{t-\tau})\right]$
- **Cross-sectional momentum** (long position in one asset and short position in the other)
  $D^A_{c,t} = \tanh\left\{\beta_c[(P^A_t - P^A_{t-\tau}) - (P^B_t - P^B_{t-\tau}]\right\}$ and
  $D^B_{c,t} = \tanh\left\{\beta_c[(P^B_t - P^B_{t-\tau}) - (P^A_t - P^A_{t-\tau}]\right\}$

ECONOMIC MEANING OF INGREDIENTS

- F_t^i and P_t^i are the (log) fundamental and the (log) price of market i at time t
- $\beta_f > 0$ measures the mean-reverting of the market price to the fundamental price
- $\beta_a > 0$ and $\beta_c > 0$ are extrapolation rates on future price trend of absolute and cross-sectional momentum traders
- $\alpha_f^i, \alpha_a^i, \alpha_c^i \in (0, 1)$ are the market$-i$ fractions of traders
- $\mu^i > 0$ speed of the price adjustment by market maker
- $\gamma_f^i = \mu^i \alpha_f^i \beta_f$, $\gamma_a^i = \mu^i \alpha_a^i \beta_a$, $\gamma_c^i = \mu^i \alpha_c^i \beta_c$ measure the activities of the three types of agents

We expand the *tanh* by using Taylor series stopping at the second order and we discretize the model by using **Nearly Exact Discretization Scheme (NEDS)**:

- After same algebra and ignoring third order terms we get:
$$\frac{dP^i}{dt} = (\gamma_a^i + \gamma_c^i - \gamma_f^i)P_t^i - (\gamma_a^i + \gamma_c^i)P_{t-1}^i - \gamma_c^i P_t^j + \gamma_c^i P_{t-1}^j + \gamma_a^i F^i - \mu^i \beta_f^3 P_t^i (F^i)^2 + \mu^i \beta_f^3 F^i (P_t^i)^2$$
- The left-hand side of the ODE is discretized as:
$$\frac{dx}{dt} \rightarrow \frac{x_{t+1} - x_t}{\phi(h)}$$
where h is the step-size and $\phi(h) = h + O(h^2)$ *as* $h \rightarrow 0^+$

- The resulting discretized equation can be written as:

$$\frac{x_{t+1} - x_t}{\phi(h)} = rx_t + g(x_t)x_t + b$$

where $r > 0$ and $g(x)$ a function which takes as input only real values

- We define $r^i = \gamma_a^i + \gamma_c^i - \gamma_f^i - \mu^i \beta_f^3 (F^i)^2$ and apply to our ODE:

$$\phi^i(h) = \frac{\exp(r^i h) - 1}{r^i}, r^i, h \neq 0$$

- **Resulting discretized system:** $P_{t+1}^i = (r^i \phi^i(h) + 1) P_t^i + \phi^i(h) \left[-(\gamma_a^i + \gamma_c^i) P_{t-1}^i - \gamma_c^i P_t^j + \gamma_c^i P_{t-1}^j + \gamma_c^i F^i + \mu^i \beta_f^3 (P_t^i)^2 \right]$

Define **deviations from fundamental prices**: $x_t^i = p_t^i - F^i$

$$\Downarrow$$

$$\{i, j = A, B\} \cap \{i \neq j\}$$

$$x_{t+1}^i =$$
$$\phi^i(h)\mu^i\beta_f^3 F^i (x_t^i)^2 + \left[r^i\phi^i(h) + 1 + 2\phi^i(h)\mu^i\beta_f^3 (F^i)^2\right] x_t^i -$$
$$\phi^i(h)\left\{(\gamma_a^i + \gamma_c^i)x_{t-1}^i + \gamma_c^i(x_t^j - x_{t-1}^j) - F^i\left(\mu^i\beta_f^3 (F^i)^2 - \gamma_c^i + r^i\right)\right\}$$

Definition

The **fundamental steady state** is defined by $(x_F^A, x_F^B) = (0, 0)$ or, equivalently, $(P_F^A, P_F^B) = (F^A, F^B)$

The fundamental price vector $(x_F^A, x_F^B) = (0, 0)$ is a fixed point if and only if $\gamma_a^i = \gamma_f^i \; \forall i = A, B$

$$\Downarrow$$

$$x_{t+1}^i = \phi^i(h)\mu^i\beta_f^3 F^i (x_t^i)^2 + \left[r^i\phi^i(h) + 1 + 2\phi^i(h)\mu^i\beta_f^3(F^i)^2\right] x_t^i - \phi^i(h)(\gamma_a^i + \gamma_c^i)x_{t-1}^i - \phi^i(h)\gamma_c^i(x_t^j - x_{t-1}^j)$$

Measures of activities are the same for absolute momentum and fundamental investors

The system admits up to four fixed points

- The **fundamental** equilibrium: $(x_F^A, x_F^B) = (0, 0)$, for any range of the parameter values
- The *A*−**fundamental** equilibrium:
  $(x_F^A, x_\star^B) = \left(0, -F^B + \frac{\gamma_a^B}{\mu^B \beta_f^3 F^B}\right)$, if $F^B \neq \sqrt{\frac{\gamma_a^B}{\mu^B \beta_f^3}}$
- The *B*−**fundamental** equilibrium:
  $(x_\star^A, x_F^B) = \left(-F^A + \frac{\gamma_a^A}{\mu^A \beta_f^3 F^A}, 0\right)$, if $F^A \neq \sqrt{\frac{\gamma_a^A}{\mu^A \beta_f^3}}$
- The **non-fundamental** equilibrium:
  $(x_\star^A, x_\star^B) = \left(-F^A + \frac{\gamma_a^A}{\mu^A \beta_f^3 F^A}, -F^B + \frac{\gamma_a^B}{\mu^B \beta_f^3 F^B}\right)$, if $F^A \neq \sqrt{\frac{\gamma_a^A}{\mu^A \beta_f^3}}$ and $F^B \neq \sqrt{\frac{\gamma_a^B}{\mu^B \beta_f^3}}$

STABILITY ANALYSIS: we introduce the auxiliary variables y_t^i for obtaining a ***system T of first-order difference equations***

$$T(x_t^A, x_t^B, y_t^A, y_t^B) = (F_A(x_t^A, x_t^B, y_t^A, y_t^B), F_B(x_t^A, x_t^B, y_t^A, y_t^B), G_A(x_t^A), G_B(x_t^B))$$

$$\begin{cases} x_{t+1}^A = \phi^A(h)\mu^A\beta_f^3 F^A(x_t^A)^2 + \left[\gamma_c^A\phi^A(h) + 1 + \phi^A(h)\mu^A\beta_f^3(F^A)^2\right] x_t^A \\ \qquad -\phi^A(h)(\gamma_a^A + \gamma_c^A)y_t^A - \phi^A(h)\gamma_c^A(x_t^B - y_t^B) \\ x_{t+1}^B = \phi^B(h)\mu^B\beta_f^3 F^B(x_t^B)^2 + \left[\gamma_c^B\phi^B(h) + 1 + \phi^B(h)\mu^B\beta_f^3(F^B)^2\right] x_t^B \\ \qquad -\phi^B(h)(\gamma_a^B + \gamma_c^B)y_t^B - \phi^B(h)\gamma_c^B(x_t^A - y_t^A) \\ y_{t+1}^A = x_t^A \\ y_{t+1}^B = x_t^B \end{cases}$$

Let $Tr(x_A, x_B)$ and $Det(x_A, x_B)$ be trace and determinant of the Jacobian matrix at a f.p. $(x_A, x_B) \Rightarrow$ **STABILITY CONDITIONS**:

$$\begin{cases} 1 - Tr(x_A, x_B) + Det(x_A, x_B) > 0 \\ 1 + Tr(x_A, x_B) + Det(x_A, x_B) > 0 \\ 1 - Det(x_A, x_B) > 0 \end{cases}$$

A **bifurcation** occurs when one of them is violated

(qualitative change in the number of equilibria or in their stability when parameters pass certain values)

In our case determinant is constant and does not depend on the coordinates of the fixed point

About the **NUMBER OF FIXED POINTS**:

- for $F^i = \sqrt{\frac{\gamma_a^i}{\mu^i \beta_f^3}}$ the $j-$fundamental equilibrium collapses with the fundamental one, while the non-fundamental equilibrium collapses with the $i-$fundamental, $\forall i, j = A, B$ and $i \neq j$
- for $F^i = \sqrt{\frac{\gamma_a^i}{\mu^i \beta_f^3}}$ $\forall i = A, B$ the fundamental steady state is unique (four coincident fixed points)

BUT when the value $F^i = \sqrt{\frac{\gamma_a^i}{\mu^i \beta_f^3}}$ is passed, the number of steady states returns as before

About the **STABILITY** of fixed points:

The fundamental steady state $(0,0)$ satisfies:

1. If $1 - Det > 0$ then $1 - Tr(0,0) + Det < 0$
2. If $1 - Tr(0,0) + Det > 0$ then $1 - Det < 0$

$\Rightarrow$ it is unstable for any set of the parameter values

Coexisting locally stable fixed points

1. $\frac{1}{\mu^i \beta_f^3}\left(\frac{Tr(0,0)-2}{2\phi^i(h)} + \gamma_a^i\right) < (F^i)^2 < \frac{1}{\mu^i \beta_f^3}\left(\frac{Tr(0,0)+2}{2\phi^i(h)} + \gamma_a^i\right)$
   $\forall i = A, B$
2. $1 - Det > 0$

$\Rightarrow$ $i-$fundamental equilibria are both locally stable

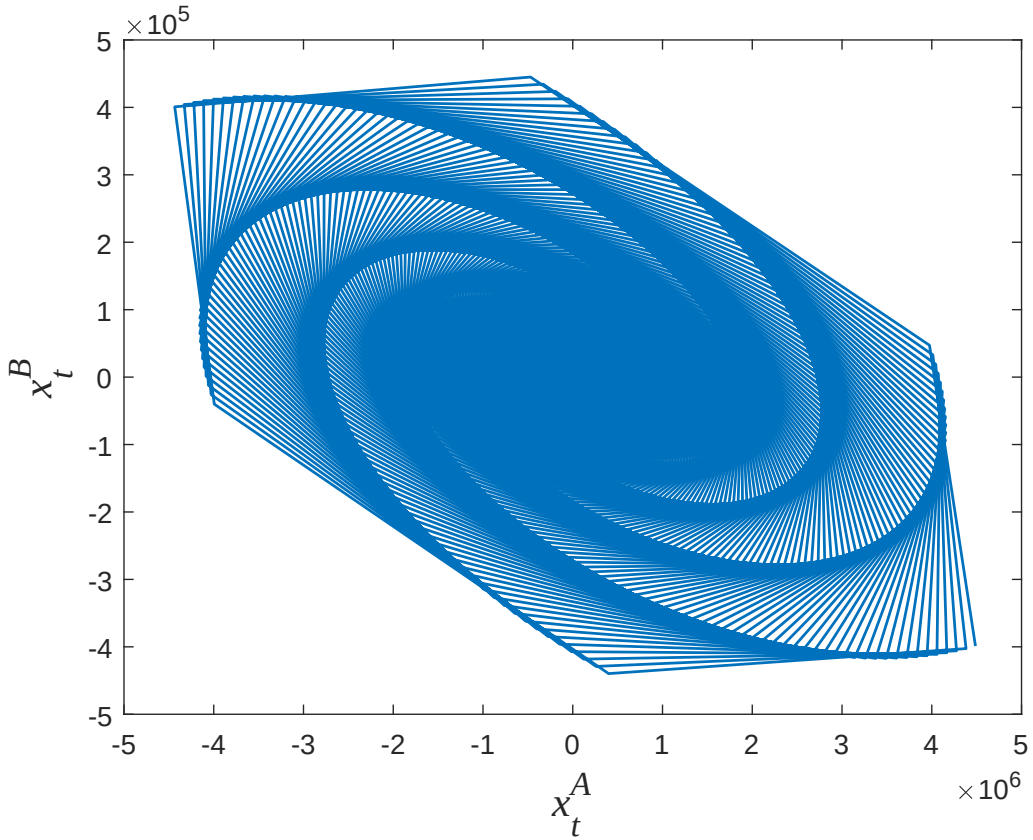

Figure: Attractor in the plane (x_t^A, x_t^B)

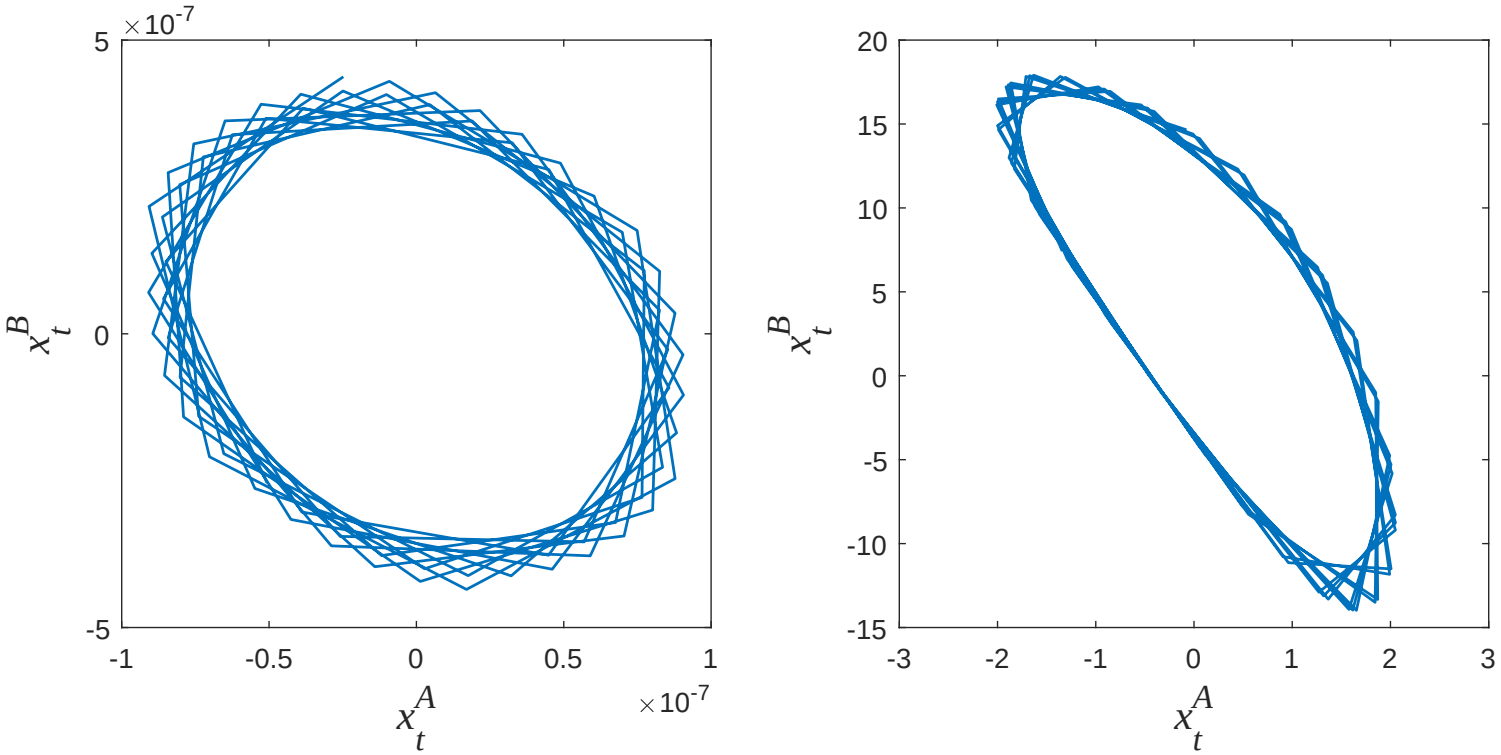

Figure: Coexisting attractors in the plane (x_t^A, x_t^B)

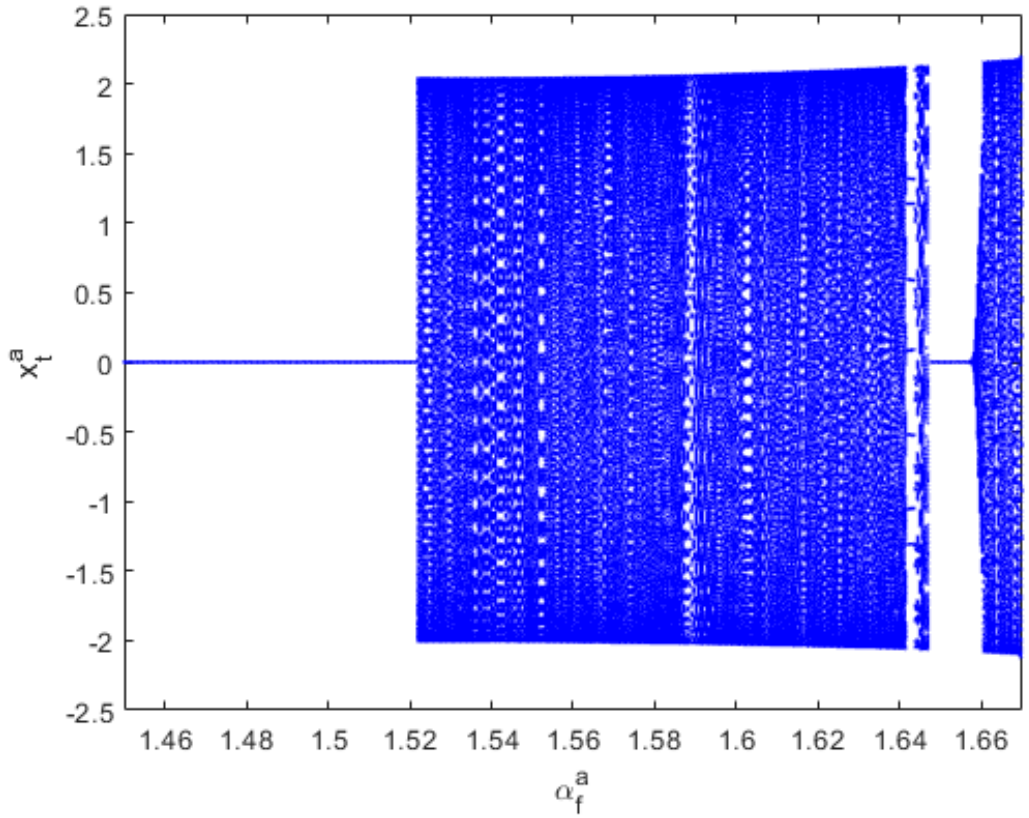

Figure: Bifurcation diagram of x_t^A w.r.t. α_f^A, initial condition close to **non-fundamental f.p.**

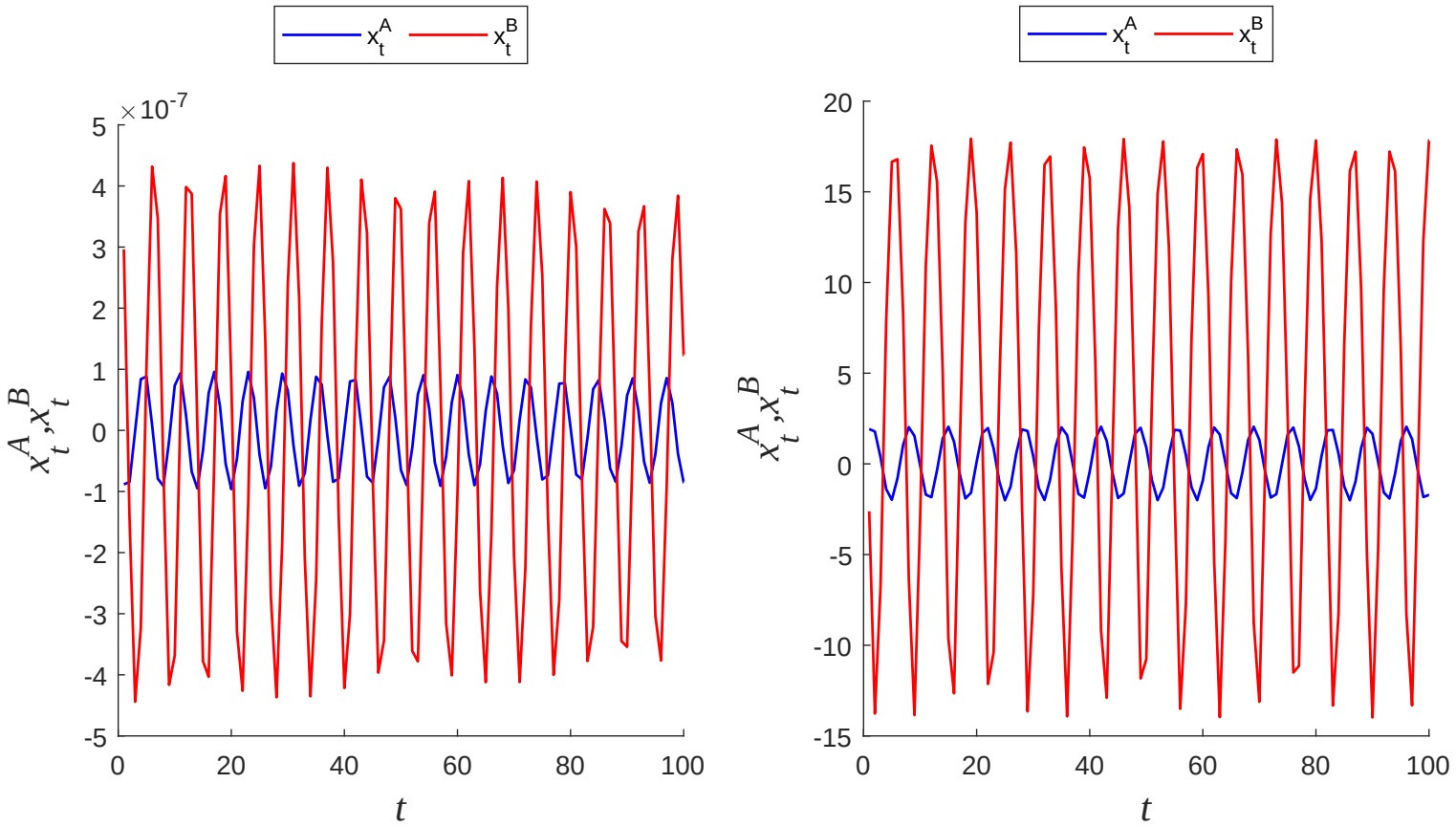

Figure: Time series of prices with i.c. close to $B-$ and $A-$fundamental, respectively

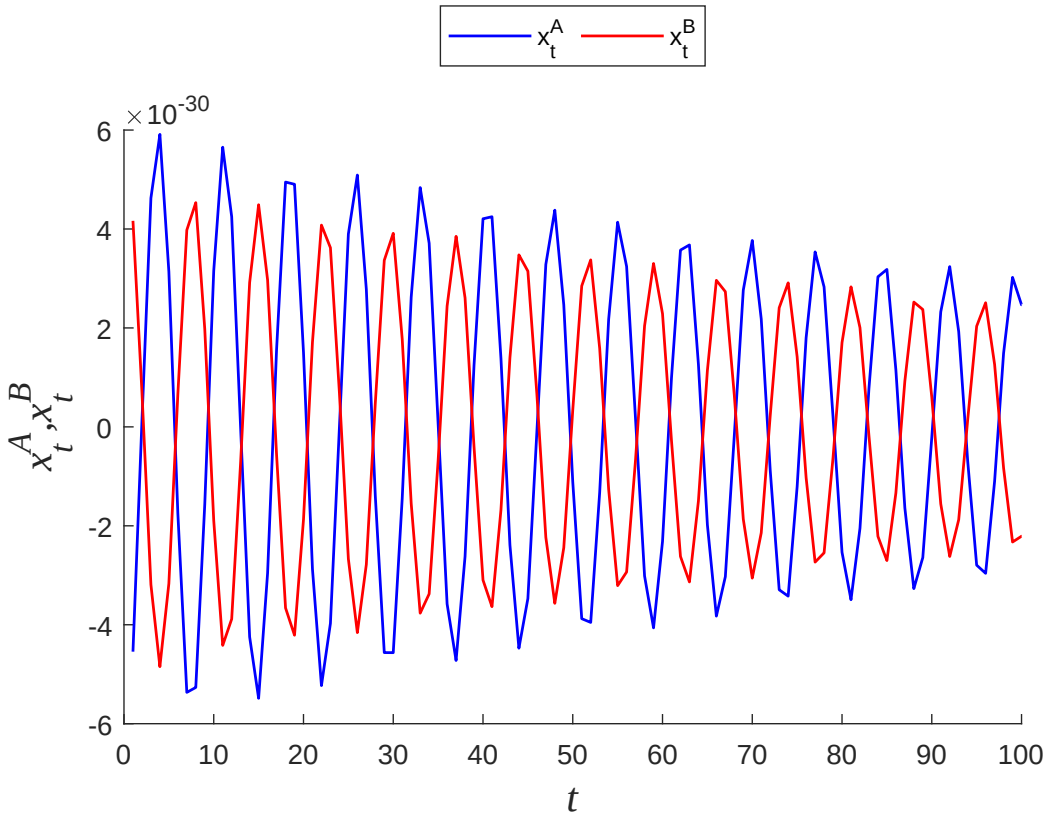

Figure: Time series of prices

TIME SERIES show:

- The amplitude of fluctuations is different, depending on i.c.
- Time series are aperiodic
- Fluctuations are larger in market B than in A: this is justified by the low $B-$fundamental value and the high value of the reaction parameter of fundamentalists in market $B \Rightarrow$ in market B they believe that the fundamental value is undervalued and buy assets
- Last figure: when in one market the trajectory is increasing then it is decreasing in the other $\Rightarrow$ an important delay in the flow of information between two markets

We are working in order to:

- further investigate the capability of the model to replicate the main **stylized facts observed in financial markets** (volatility clustering, fat tails and long memory effects)
- test our model **for forecasting returns** relying on a machine learning technique

  $\mapsto$ we rely on **Random Forests** which is able to capture the non-linearity of the model appropriately

Thank you for the attention!

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