Demand forecasting under lost-sales stock policies with Tobit Exponential Smoothing

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Outline

Introduction

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In general, stock-outs are modelled on the basis of two extreme cases:

- Back ordering: customers are allowed to place orders when there is not sufficient stock on hand. No demand is lost, but placed at a different time.
- Lost sales: excess demand over the inventory hold by the company is simply lost. Demand is not equal to sales in case of stock outs.

Introduction

In the case of lost sales, demand is underestimated and it will result in lower inventory level and a lower customer service level, which may cause a spiral-down effect. The problem of estimating demand based on sales is also known in the statistics literature as censored estimation,

The problem

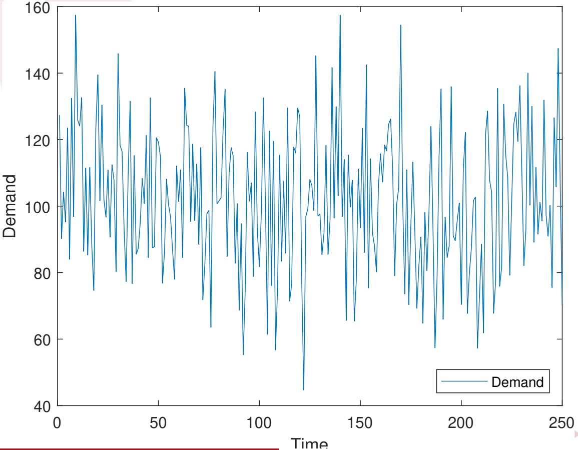
Let y_t and y_t^* the sales and demand, respectively, of a determined product at time t, with $t = 1, \ldots, n$. If $Y_{max,t}$ is the upper censored level which can vary with time, only when $y_t < Y_{max,t}$ the true demand is observed (Type I censoring), such as:

$$y_t = \begin{cases} y_t^*, & y_t^* < Y_{max,t} \\ Y_{max,t} & y_t^* \geq Y_{max,t} \end{cases} \tag{1}$$

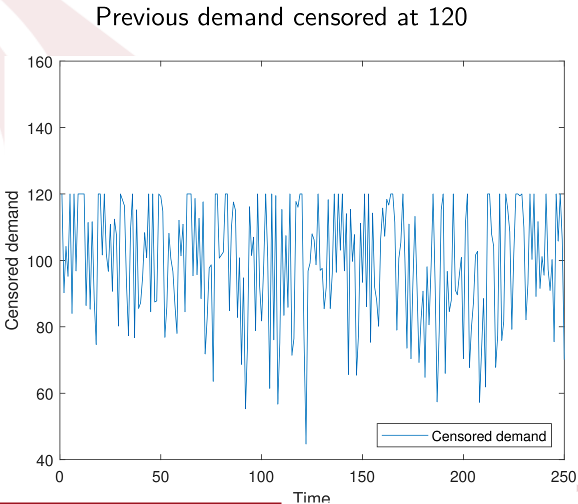
The problem is how to forecast y_t^* when we only have available n observations of y_t and $Y_{max,t}$.

The problem

Simulated random Gaussian demand (mean 100 and variance 400)

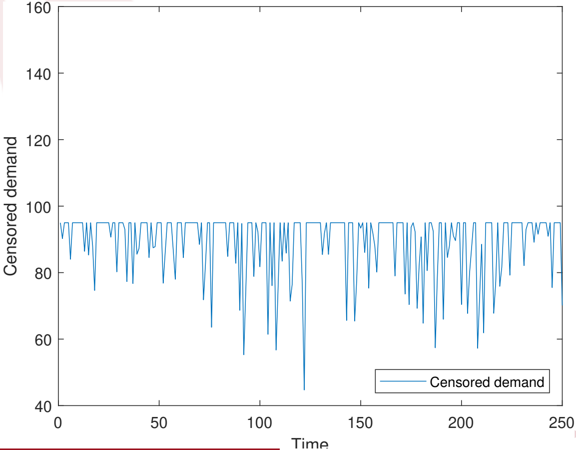


The problem


Previous demand censored at 120
Censored demand
160
140
120
100
80
60
40
0
50
100
150
200
250
Time
Censored demand

The problem

Previous demand censored at 95



Solutions

There is a local solution to this problem since the 90's: S. Nahmias (1994), Demand estimation in lost sales inventory systems, Nav.459 Res. Logist. 41, 739–757.

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Inconveniences:

- Analytically developed under specific strong assumptions.
- Assume constant censored level.
- Works reasonably when the censored level is above the mean level.

Solutions

A far more general and elegant solution is the Tobit Kalman Filter: B. Allik (2014), The Tobit Kalman Filter: An estimator for censored data,465 Ph.D. thesis, University of Delaware.

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- It works on a general form of State Space models, therefore, many different dynamic forecasting models may be implemented. For example, models may include trends, seasonal and other components.

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- It works with time varying censored levels. As a matter of fact, the censoring level depends on previous forecasts themselves.
- The assumptions are less strict and works in situations with severe censoring.

Solutions

Standard State Space model:

$$\begin{aligned} \alpha_{t+1} &= T\alpha_t + G\eta_t, \quad & \eta_t \sim N(0, Q) \\ y_t &= Z\alpha_t + \epsilon_t, \quad & \epsilon_t \sim N(0, H) \end{aligned}$$

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y_t &= \begin{cases} y_t^*, & y_t^* < Y_{max,t} \\ Y_{max,t} & y_t^* \geq Y_{max,t} \end{cases}
\end{aligned}
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Solutions

Standard Kalman Filter:

Prediction step:

$$a_{t+1} = Ta_{t|t}, \qquad P_{t+1} = TP_{t|t}T' + GQG',$$

Update step:

$$\mathrm{E}(y_t|a_t) = Za_t$$

$$
\begin{aligned}
v_t &= y_t - \mathrm{E}(y_t|a_t), & R_t^* &= \sigma^2 \\
R_{1,t} &= P_t Z' & R_{2,t} &= ZR_{1,t} + R_t^*, \\
K_t &= R_{1,t}R_{2,t}^{-1}, & \sigma^2 &= ZP_tZ' + H, \\
a_{t|t} &= a_t + K_t v_t, & P_{t|t} &= P_t - K_t R_{1,t}',
\end{aligned}
$$

Solutions

Tobit Kalman Filter:

Prediction step:

$$a_{t+1} = Ta_{t|t}, \qquad P_{t+1} = TP_{t|t}T' + GQG',$$

Update step:

$$\mathrm{E}(y_t|a_t) = P_{un,t}(Za_t - \sigma l_t) \quad + P_{max,t}Y_{max,t},$$

$$v_t = y_t - \mathrm{E}(y_t|a_t), \qquad R^*_t = \sigma^2(1 - P^{-1}_{un,t}c_t - l^2_t),$$

$$R_{1,t} = P_tZ'P_{un,t}, \qquad R_{2,t} = P_{un,t}ZR_{1,t} + R^*_t,$$

$$K_t = R_{1,t}R^{-1}_{2,t}, \qquad \sigma^2 = ZP_tZ' + H,$$

$$a_{t|t} = a_t + K_tv_t, \qquad P_{t|t} = P_t - K_tR'_{1,t},$$

Solutions

Additional terms necessary:

$$P_{max,t} = 1 - \Phi\left(\frac{Y_{max,t} - Za_t}{\sigma}\right)$$

$$P_{un,t} = 1 - P_{max,t},$$

$$I_t = P_{un,t}^{-1}\phi\left(\frac{Y_{max,t} - Za_t}{\sigma}\right),$$

$$c_t = \frac{Y_{max,t} - Za_t}{\sigma}\phi\left(\frac{Y_{max,t} - Za_t}{\sigma}\right).$$

where ϕ and Φ are the standard Gaussian probability and cumulative density functions, respectively.

Solutions

Innovations form State Space model (single source of noise):

$$\begin{aligned} \alpha_t &= T\alpha_{t-1} + G\epsilon_t, \\ y_t &= Z\alpha_{t-1} + \epsilon_t, \qquad \epsilon_t \sim N(0, H) \end{aligned}$$

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The advantage of innovations form is that the states are known with no uncertainty as a direct function of data. Given an initial state vector α_0 the states and the perturbation may be estimated recursively, i.e.,

$$\begin{aligned} \epsilon_t &= y_t - Za_{t-1}, \\ a_t &= Ta_{t-1} + G(y_t - Za_{t-1}), \end{aligned}$$

Solutions

Then, there is no need of a Kalman filter or the Kalman filter collapses to the previous equations.

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Typically ExponenTial Smoothing and ARIMA models may be cast in this form.

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It can be shown that the optimal recursions for this model are

$$
\begin{aligned}
\mathrm{E}(y_t|a_{t-1}) &= P_{un,t}(Za_{t-1} - \sigma l_t) + P_{max,t}Y_{max,t} \\
\epsilon_t &= y_t - \mathrm{E}(y_t|a_{t-1}) \\
\mathrm{E}(a_t|Y_t) &= Ta_{t-1} + P_{un,t}/(1 + c_t/P_{un,t} - l_t^2)G\epsilon_t,
\end{aligned} \tag{2}
$$

Solutions

Estimation of model parameters is carried out by maximum likelihood including the initial state vector ($|\alpha_0$) as known fixed values. Taking ϕ as the Gaussian standard pdf, the likelihood function for the Tobit model is

$$L(y_1, \ldots, y_n) = \prod_{y_t = Y_{max,t}} P_{max,t} \times \prod_{y_t < Y_{max,t}} \frac{1}{\sigma} \phi \left(\frac{y_t - Za_{t-1}}{\sigma} \right). \tag{3}$$

Example 1: Random Gaussian demand

Simple Exponential Smoothing model (SES):

$$
\begin{aligned}
l_t &= l_{t-1} + \alpha\epsilon_t, \\
y_t &= l_{t-1} + \epsilon_t, \qquad \epsilon_t \sim N(0, H) \\
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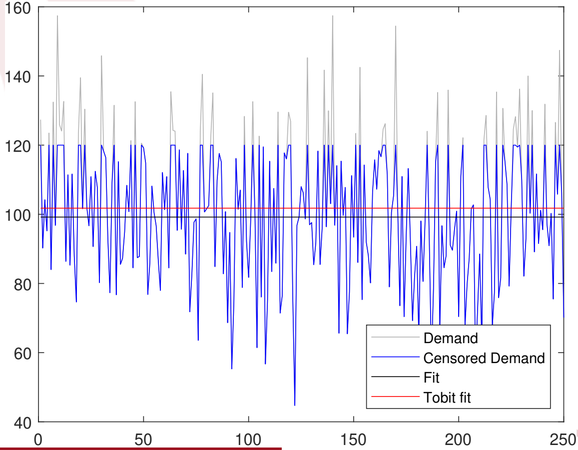
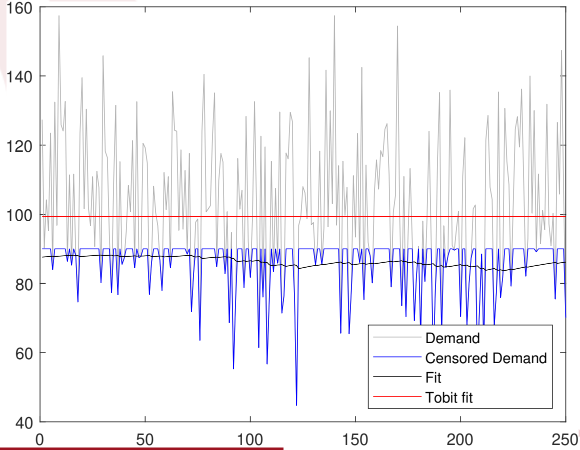
Example 1: Random Gaussian demand

Simulated random Gaussian demand Censored at 120.



Example 1: Random Gaussian demand

Simulated random Gaussian demand Censored at 90.



Example 2: Model with trend and seasonality

Holt-Winters model (HW):

$$
\begin{aligned}
l_t &= l_{t-1} + b_{t-1} + \alpha \epsilon_t, \\
b_t &= b_{t-1} + \beta \epsilon_t, \\
s_t &= s_{t-m} + \gamma \epsilon_t, \\
y_t &= l_{t-1} + b_{t-1} + s_{t-m} + \epsilon_t, \qquad \epsilon_t \sim N(0, H) \\
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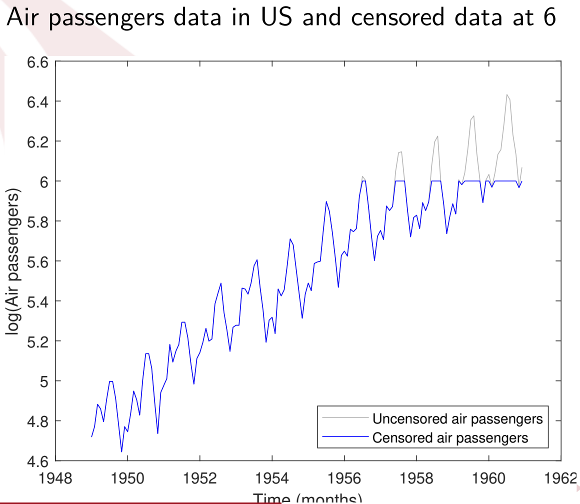
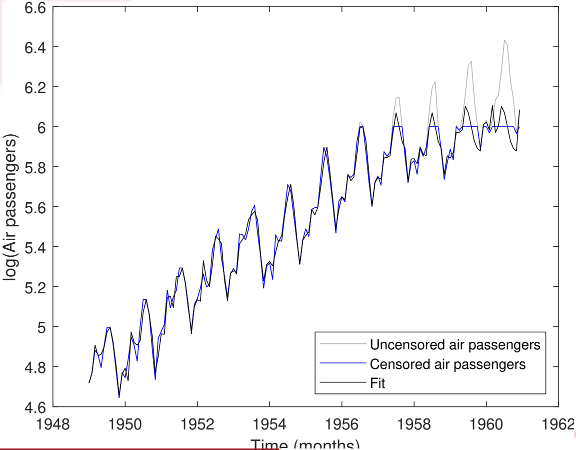


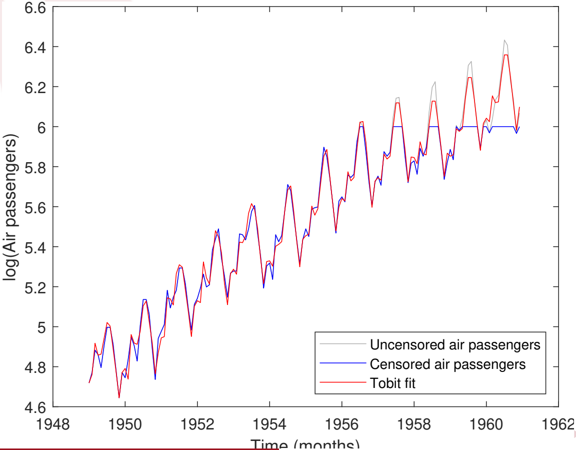
Example 2: Model with trend and seasonality

Air passengers data in US, censored data at 6 and standard fit



Example 2: Model with trend and seasonality

Air passengers data in US, censored data at 6 and Tobit fit



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- Previous solutions are local.
- Tobit State Space combined with the Tobit Kalman filter provides an elegant and general solution.
- Such solution is in fact a general framework in which many differentt dynamic forecasting models fit in.

Thank you for your attention!

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blogs: blog.uclm.es/diegopedregal/
blog.uclm.es/juanramontrapero/

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