A semi-Markov approach to financial modelling during the COVID19 pandemic

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Introduction

- **macro-to-micro** approach: application of the econometric tools of the time series analysis
  the observed price is considered to be a noisy representation of an unobserved price
- **micro-to-macro** approach: directly models the observable quantities.
  The focus is only on the observable quantities and it exploits the point processes (Fodra and Pham, 2015)

Literature

- First attempt of modelling the financial time series using a semi-Markov chain (D'Amico and Petroni, 2012a)
- Extension of the model with the introduction of a memory index (D'Amico and Petroni, 2011)
- Best results with the weighted-indexed semi-Markov chain (WISMC) (D'Amico and Petroni, 2012b)
- Extension to multivariate settings (D'Amico and Petroni, 2018, 2021)

The WISMC model has proven to be able to reproduce important stylised facts of financial time series such as the first-passage-time distributions and the persistence of volatility.

Markov renewal process

Let us consider a finite state space E and a probability space $(\Omega, \mathcal{F}, P)$. The two random variables

$$X_n : \Omega \to E \qquad T_n : \Omega \to \mathbb{R}_+,$$

with $n \in \mathbb{N}$ and $0 = T_0 < T_1 < T_2 < \dots$, form a *Markov renewal process* (X, T) with state space $E \times \mathbb{R}^+$ if

$$\begin{aligned} &\mathbb{P}(X_{n+1} = j, T_{n+1} - T_n \leq t | X_0, \dots, X_n; T_0, \dots, T_n) \\ &= \mathbb{P}(X_{n+1} = j, T_{n+1} - T_n \leq t | X_n), \quad a.s. \quad \forall n \in \mathbb{N}, j \in E, t \in \mathbb{R}_+. \end{aligned} \tag{1}$$

Semi-Markov process (1/2)

Assuming that the process is temporally homogeneous, the probability

$$\mathbb{P}(X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i) = Q(i, j, t) \tag{2}$$

is independent of n, and Q is called a *semi-Markov kernel*. In general, $Q(i, j, 0) = 0$, $\forall i, j \in E$.
For each pair (i, j),

$$\lim_{t \longrightarrow +\infty} Q(i, j, t) = P(i, j), \tag{3}$$

with $P(i, j) \geq 0$ and $\sum_{j \in E} P(i, j) = 1, i, j \in E$. The quantities $P(i, j)$ are the transitions probabilities of the Markov chain $\{X_n\}_{n \in N}$ with state space E.

semi-Markov process (2/2)

The *semi-Markov process* with state space E and transition kernel $Q(i, j, t)$ is the continuous-time parameter process

$$Y_t = X_n \text{ for } t \in [T_n, T_{n+1}). \tag{4}$$

This process can be seen as the state at time t of some system which moves from one state to another with random sojourn times in between (Çinlar, 1975). The sojourn interval $[T_n, T_{n+1})$ represents a random variable with a distribution which depends on the state being visited X_n and the next state to be visited X_{n+1}.

Memory index

Let U_n be a stochastic process with values in $\mathbb{R}$. This random variable represents the index process which stores the historical information of the semi-Markov process and can be expressed as in D'Amico and Petroni (2021)

$$U_n(\boldsymbol{\theta}) = \sum_{k=0}^{n-1} \sum_{a=T_{n-1-k}}^{T_{n-k}-1} f(X_{n-1-k}, T_n, a, \boldsymbol{\theta}) + f(X_n, T_n, T_n), \quad (5)$$

where $f : E \times \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is a Borel measurable bounded function and U_0 is known and non-random. The function f depends on the current time a, on the state X_{n-1-k} visited at current time and on the vector of parameters, $\boldsymbol{\theta}$, that is specific to the weighting process.

WISMC

For the WISMC model we can then make the following assumption on the process

$$\begin{aligned}\mathbb{P}(X_{n+1} = j, T_{n+1} - T_n \leq t | X_0, \dots, X_n; T_0, \dots, T_n; U_0, \dots, U_n) \\ = \mathbb{P}(X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i, U_n = v) := Q(i, j, t, v),\end{aligned} \tag{6}$$

on the transition probabilities

$$P(i, j, v) = \mathbb{P}(X_{n+1} = j | X_n = i, U_n = v), \tag{7}$$

and on the conditional waiting time distribution function

$$G(i, j, t, v) = \mathbb{P}(T_{n+1} - T_n \leq t | X_n = i, X_{n+1} = j, U_n = v). \tag{8}$$

Model calibration

Issues with application to financial data:

- Price returns are continuous
- Memory index values are continuous

D'Amico and Petroni (2012b) rely on an arbitrary discretisation based on the visual observation of the histograms.

Discretisation algorithms:

- quantile
- sigma
- k-means
- Gaussian mixture model (GMM)

Discretisation algorithms

- **quantile**: splits the observations in k quantiles, where k is the number of states
- **sigma** (De Blasis, 2020):

State 1	State 2	State 3	State 4
$[min, -\sigma)$	$[-\sigma, 0)$	$[0, \sigma)$	$[\sigma, max]$

State 1	State 2	State 3	State 4	State 5
$\left[min, -2\frac{\sigma}{2}\right)$	$\left[-2\frac{\sigma}{2}, -\frac{\sigma}{2}\right)$	$\left[-\frac{\sigma}{2}, \frac{\sigma}{2}\right)$	$\left[\frac{\sigma}{2}, 2\frac{\sigma}{2}\right)$	$\left[2\frac{\sigma}{2}, max\right]$

- **k-means** (Sebestyen, 1962, MacQueen, 1967):
partitions the observations to minimise the within-cluster sum of squares using an iterative algorithm
- **GMM** (Wolfe, 1963, Scott and Symons, 1971):
based on the assumption that the observations are generated by a mixture of Gaussian distributions with unknown parameters.

Function of the memory index

D'Amico and Petroni (2012b) adopted the exponentially weighted moving average (EWMA) of the squares of the returns X_n as the function of the index U_n. Using the EWMA, the vector $\boldsymbol{\theta}$ reduces to the single smoothing parameter λ and the function f becomes

$$f(X_{n-1-k}, T_n, a, \lambda) = \frac{\lambda^{T_n - a} X_{n-1-k}^2}{\sum_{k=0}^{n-1} \sum_{a=T_{n-1-k}}^{T_{n-k}-1} \lambda^{T_n - a}}. \tag{9}$$

Employing the squared returns, the index U_n can be seen as a volatility index. So, the dynamic of the process will depend on the volatility level.

Note: in the application we fix $\lambda = 0.97$

The process U_n has values in $\mathbb{R}$ and needs to be discretised.

Monte Carlo simulation

After the returns and index discretisation, we can estimate the transition probabilities $P(i, j, v)$ and the cumulative distribution function of the sojourn times $G(i, j, t, v)$. Then, we proceed with a Monte Carlo simulation (D'Amico and Petroni, 2012b):

1. set $n = 0$, $J_0 = i$, $T_0 = 0$, $U_0 = v$, horizon time $= T$
2. given J_n and U_n, sample J from $P(i, j, v)$ and set J_{n+1}
3. given J_n and J_{n+1}, sample W from $G(i, j, t, v)$ and set $T_{n+1} = T_n + W$
4. set U_{n+1} using the index function
5. if $T_{n+1} \geq T$ stop, else set $n = n + 1$ and go to 2.

Model test

Verify if the simulated series follows the long range serial correlation of the squared returns which is typical of financial returns.

$$\Sigma(\tau) = \frac{Cov(Y^2(t+\tau), Y^2(t))}{Var(Y^2(t))}, \tag{10}$$

where Y is the WISMC process of the returns and τ is the time lag. We estimate $\Sigma(\tau)$ for the real and simulated returns and compute the RMSE between them to compare the use of different calibration strategies.

Bitcoin data

- Bitcoin prices from Binance website: open 24/7
- 1-minute Log-returns: date range from 1 March 2021 to 28 February 2022
- 1-second Log-returns: date range from 21 February 2022 to 28 February 2022

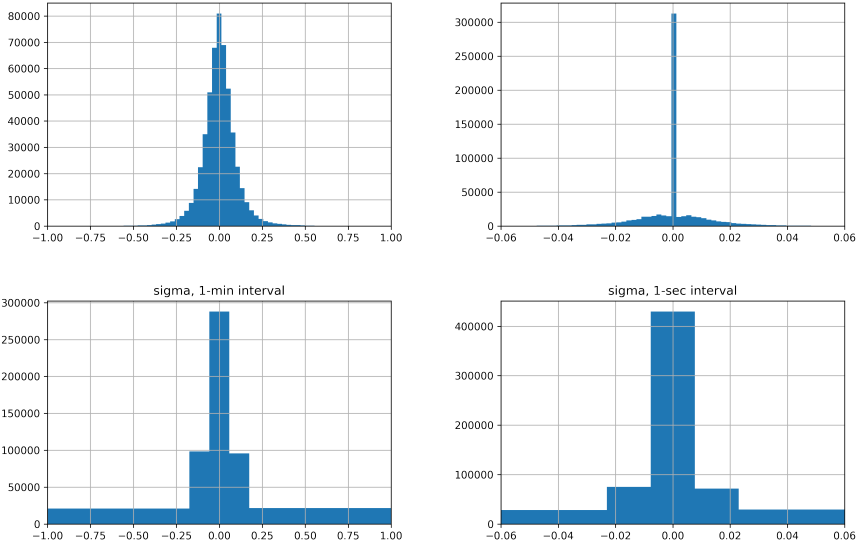
	1-minute interval		1-second interval	
	price	log-returns	price	log-returns
obs	524680		634172	
mean	47501	0.0000	38269	0.0000
std	9321	0.1160	1361	0.0153
min	28868	-5.9738	34330	-0.3996
25%	39395	-0.0519	37639	-0.0016
50%	47320	-0.0001	38435	0.0000
75%	56074	0.0507	39040	0.0008
max	69000	4.8604	44219	0.4718
skewness		-0.0266		0.3774
kurtosis		84.6502		30.4025

Returns discretisation: 5-state chain example

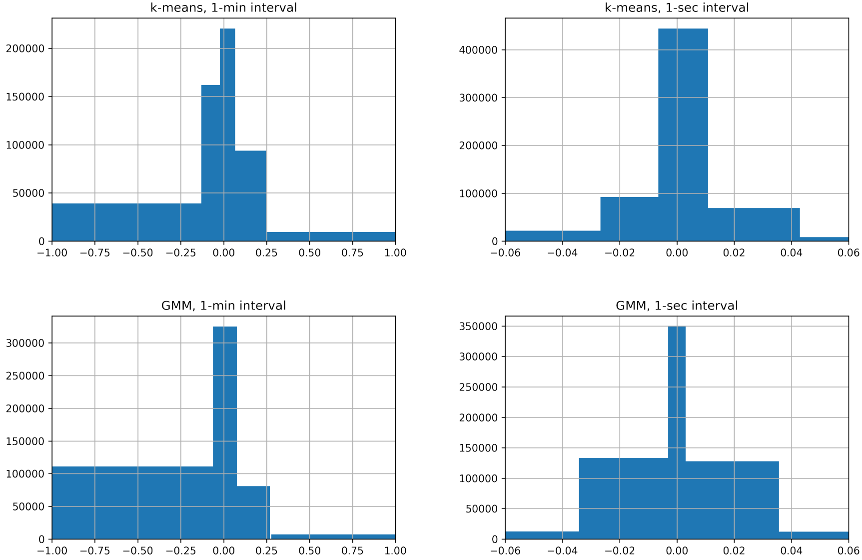
	State -2	**State -1**	**State 0**	**State 1**	**State 2**
Panel A: 1-minute interval					
quantile	[-5.974,-0.066)	[-0.066,-0.018)	[-0.018,0.017)	[0.017,0.065)	[0.065,4.860]
sigma	[-5.974,-0.174)	[-0.174,-0.058)	[-0.058,0.058)	[0.058,0.174)	[0.174,4.860]
k-means	[-5.974,-0.130)	[-0.130,-0.022)	[-0.022,0.066)	[0.066,0.247)	[0.247,4.860]
GMM	[-5.974,-0.065)	[-0.065, 0.076)	[0.076,0.261)	[0.261,0.325)	[0.325,4.860]
Panel B: 1-second interval					
quantile	[-0.4,-0.005)	[-0.005, 0.000)	[0.000,0.000)	[0.000,0.005)	[0.005,0.472]
sigma	[-0.4,-0.023)	[-0.023,-0.008)	[-0.008,0.008)	[0.008,0.023)	[0.023,0.472]
k-means	[-0.4,-0.027)	[-0.027,-0.006)	[-0.006,0.011)	[0.011,0.043)	[0.043,0.472]
GMM	[-0.4,-0.034)	[-0.034,-0.003)	[-0.003,0.003)	[0.003,0.036)	[0.036,0.472]

Note: the use of different discretisation approaches leads to different distributions, thus the WISMC processes Y_t could be in different states at the same time for each discretisation.

Returns histograms (1/1)



Returns histograms (1/2)

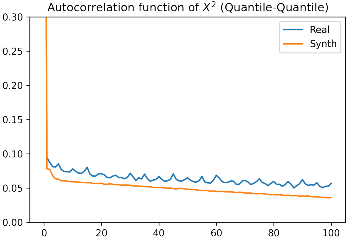


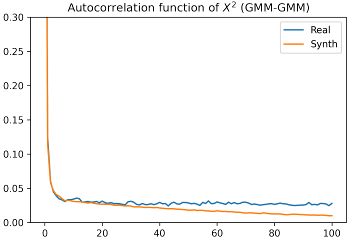
RMSE Results

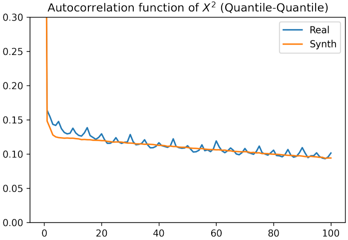
	1-minute interval			1-second interval		
Panel A: 3-state process; 3-state index						
	quantile	*k-means*	*GMM*	*quantile*	*k-means*	*GMM*
quantile	**0.015**	0.0194	0.0189	0.0385	0.0673	0.052
sigma	0.0187	0.0279	0.0291	0.044	0.0454	0.0464
k-means	0.028	0.0269	0.0309	0.0637	0.0591	0.0465
GMM	-	-	-	**0.01**	**0.0103**	**0.0103**
Panel B: 5-state process; 5-state index						
	quantile	*k-means*	*GMM*	*quantile*	*k-means*	*GMM*
quantile	**0.0065**	0.0137	0.0185	-	-	-
sigma	0.0554	0.0124	0.0312	0.0525	0.0488	0.0485
k-means	0.0332	0.0212	0.0271	0.0686	0.0461	0.0417
GMM	0.066	0.0271	0.0414	0.0725	**0.0374**	0.0468

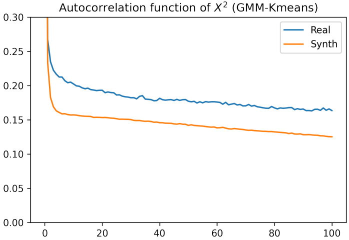
returns discretisation by rows, index discretisation by columns

ACF: 1-min (left), 1-sec (right)





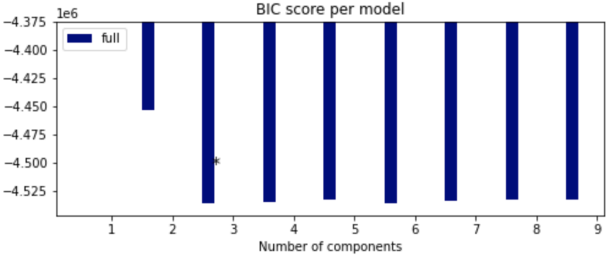




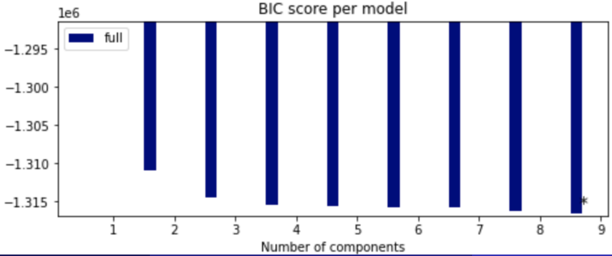
3-state,3-state (top-left), 3-state,3-state (top-right)
5-state,5-state (bottom-left), 5-state,5-state (bottom-right)

Number of states: selection criterion

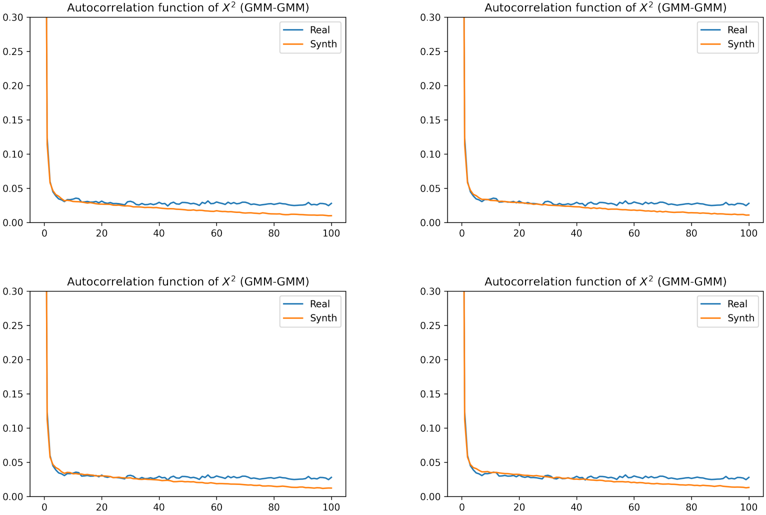
GMM returns discretisation



GMM index discretisation



ACF with BIC states selection: 1-sec intervals



3-state,3-state (top-left), 3-state,4-state (top-right)
3-state,5-state (bottom-left), 3-state,9-state (bottom-right)

Conclusion

- New calibration strategies for the WISMC model: returns and index discretisation applied to the Bitcoin market
  - Quantile-Quantile good for lower frequency data
  - GMM-GMM good for higher frequency data
- GMM permits selection of number of states via BIC comparison
- The calibrated model is able to reproduce the long-run serial correlation of the Bitcoin returns

Further development:

- Test the forecasting ability of the model
- Apply the calibration methods to other financial assets
- Propose other functions of the index computation
- Comparison with macro-to-micro approaches (e.g., GARCH)

Thank you for your attention

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