Dynamical Component Analysis: Updated and improved algorithm, applications, and limitations

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This work was supported in part by the German Federal Ministry of Education and Research (BMBF, Funding number: 05M20WBA) and in part by the European Regional Development Fund (ERDF, Project: TZM).

Goal: Low-dimensional decomposition of high-dimensional signal

- **Data**
  - **vector notation** $\mathbf{q}(t) \in \mathbb{R}^N$ **with** $t = t_1, \ldots, t_T$
  - **matrix notation** $Q \in \mathbb{R}^{N \times T}$
- **Goal: Decomposition**
  - **vector notation** $\mathbf{q}(t) = \sum_{i=1}^{n} \mathbf{w_i} x_i(t)$
  - **matrix notation** $Q = WX$ **with** $W \in \mathbb{R}^{N \times n}$, $X \in \mathbb{R}^{n \times T}$

  with $n \ll N$ **similar to**
  - **PCA (Principal Component Analysis), focusing on fast converging signal representation**
  - **ICA (Independent Component Analysis), goal: statistically independent amplitudes** $x_i(t)$
- **New algorithm, focusing on dynamics: DyCA (Dynamical Component Analysis)**

DyCA - Dynamical Component Analysis: Assumptions

Full rank (N) dataset Q

$$\underbrace{Q}_{N\times T} = \underbrace{W}_{N\times n}\underbrace{X}_{n\times T} + \underbrace{\Psi}_{N\times p}\underbrace{\Xi}_{p\times T} + \underbrace{R}_{N\times T}$$

with

- **low rank (n) deterministic signal WX**
- **low or high rank (p) stochastic components $\Psi\Xi$**
- **with independent W and Ψ**
- **small full rank residual noise R**

The deterministic amplitudes $X = [X_L\, X_{NL}]$ obey a set of differential equations with

- **m linear differential equations $\dot{X}_L = AX$ and**
- **$n - m$ non-linear equations $\dot{X}_{NL} = F(X)$**

DyCA - Dynamical Component Analysis: Algorithm

- **Introducing**
  - $U, V \in \mathbb{R}^{N \times m}$ **with** $X_L = U^T Q$ **and** $AX = V^T Q$
  - **and correlation matrices** $C_0 = QQ^T$, $C_1 = \dot{Q}Q^T$, $C_2 = \dot{Q}\dot{Q}^T$,
- **mimizing**

$$||\dot{X}_L - AX||_F^2 = ||U^T\dot{Q} - V^T Q||_F^2 \text{ s. t. } \mathbf{diag}(U^T C_2 U) = I_m$$

  with respect to U **and** V
- **leads to the generalized eigenvalue problem**

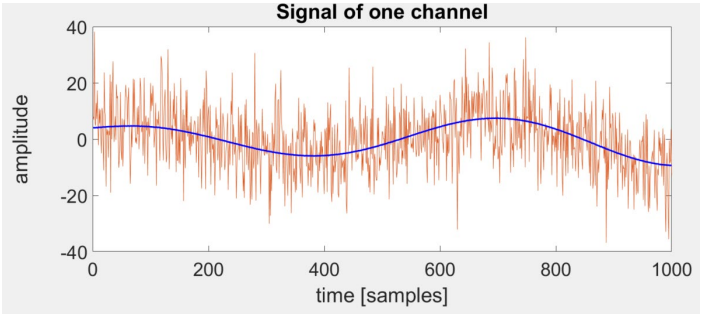
$$C_1 C_0^{-1} C_1^T U = C_2 U \Lambda, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{pmatrix}$$

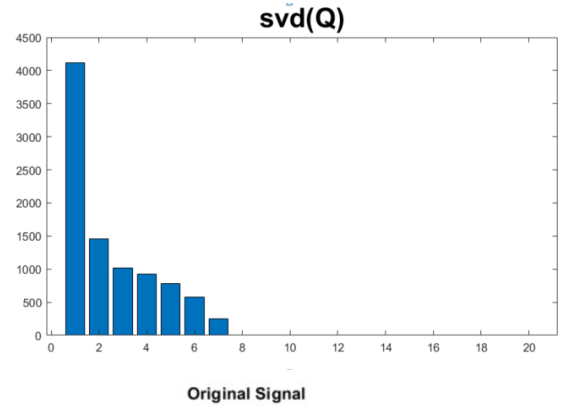
- **and** $V = C_0^{-1} C_1^T U$.
- **The generalized eigenvalues** λ_i **measure the quality of fit:** $\lambda_i = 1$**: perfect fit,** $\lambda_i = 0$**: no fit.**

Obtained Estimations (if assumptions are fulfilled)

- # of lin. eq. m by threshold α: $m = |\{\lambda_i | \lambda_i > \alpha\}|$
- subspace of deterministic amplitudes X by singular value decomposition of $[U^T Q, V^T Q]$
- # of deterministic components n by thresholding singular values (σ_i): $n = |\{\sigma_i | \sigma_i > \beta\}|$
- deterministic part of the signal $WT^{-1}TX$ - after additional optimization
- coefficients of matrix A of linear differential equation $\dot{X}_L = AX$

Simulated Example

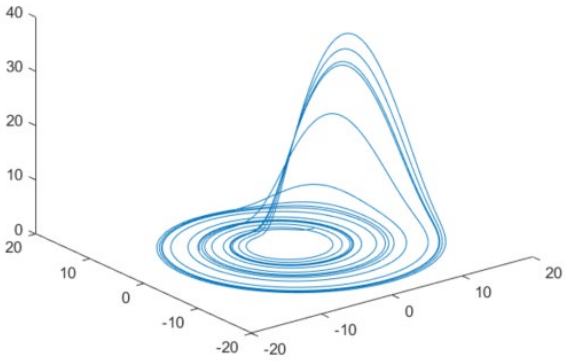


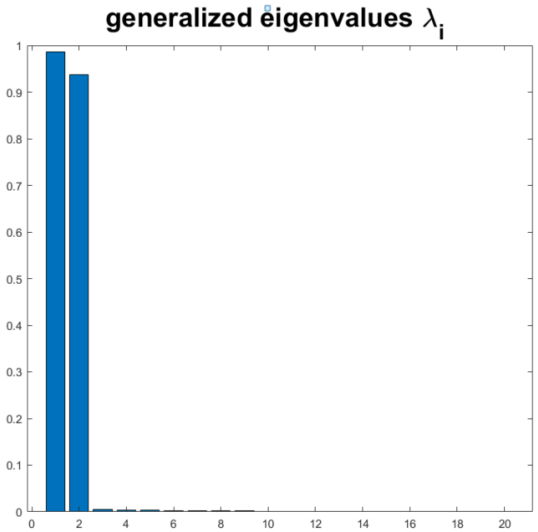


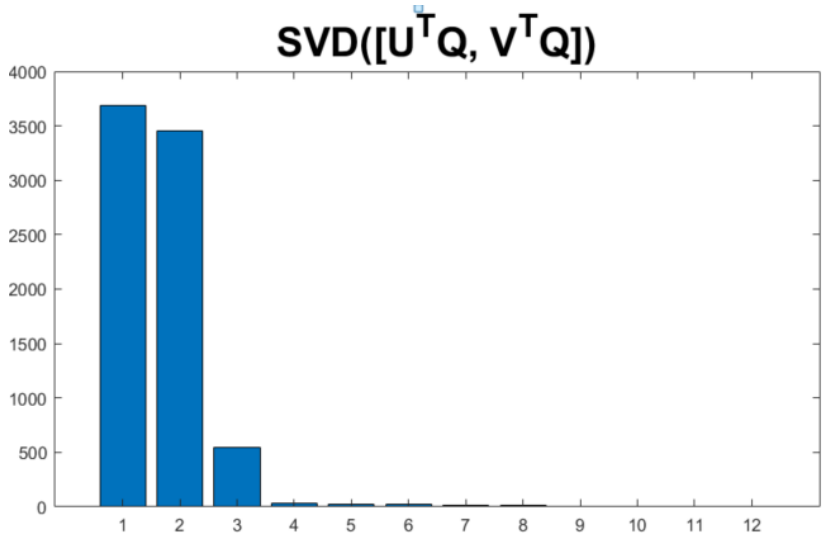
Full rank $(N = 20, T = 10000)$ **dataset** Q **with**

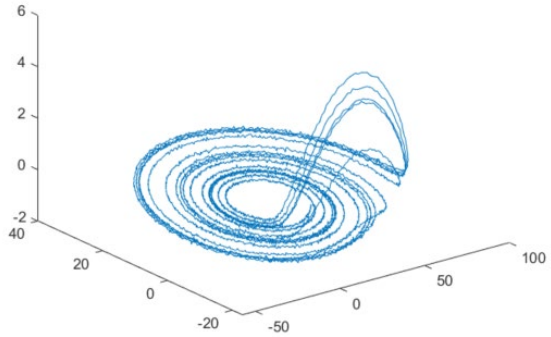
- **low rank** $(n = 3)$ **deterministic signal** WX**: Rössler attractor**
- **low or high rank** $(p = 4)$ **stochastic components** $\Psi\Xi$ **(SNR: 0dB)**
- **small full rank residual noise** R **(SNR: 50dB)**
- $m = 2$ **linear differential equations** $\dot{x}_1 = -(x_2 + x_3), \quad \dot{x}_2 = x_1 + ax_2$
- $n - m = 1$ **non-linear equations** $\dot{x}_3 = b + (x_1 - c)x_3$

Original Signal



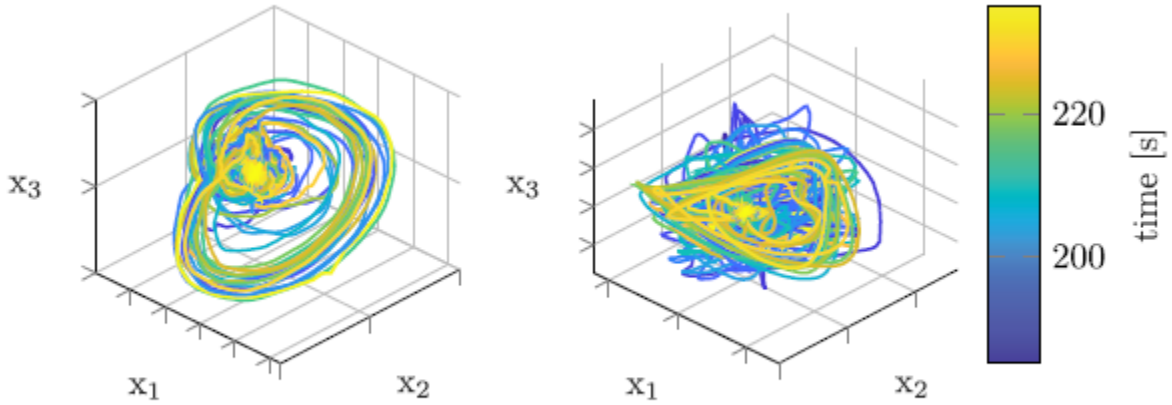






DyCA – Application: A Intracranial EEG (icEEG) data (from (2))

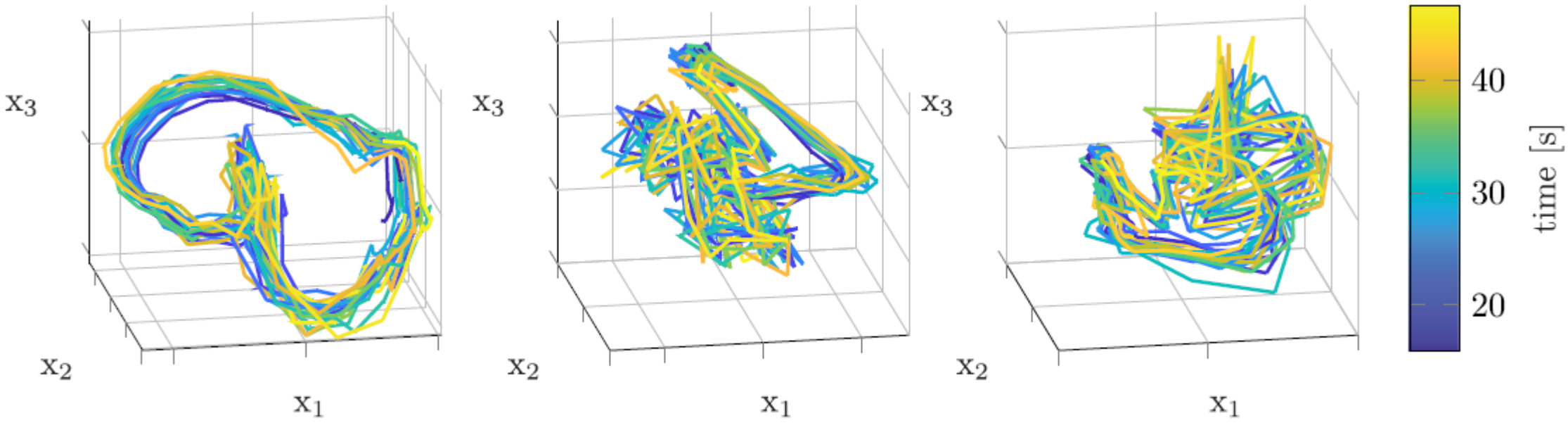
- Focal eplileptic seizure, 165 sensors, 1024 samples per second
- Two DyCA eigenvalues close to 1: λ_1=0.88, λ_2=0.86
- 3-dim dynamics



(a) DyCA (b) PCA

DyCA – Application: B Motion sense dataset (from (2))

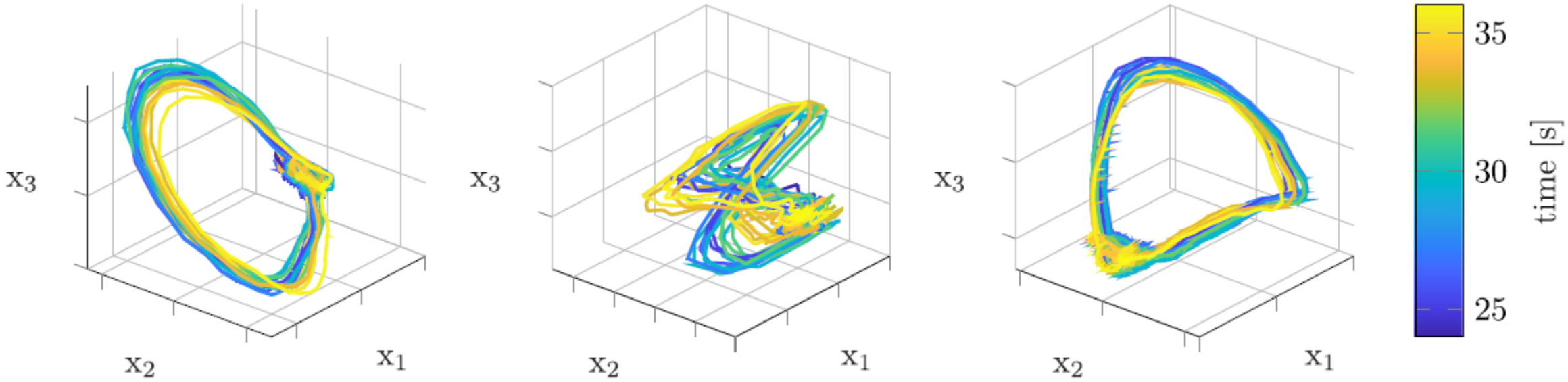
- Recording during some activities, 11 sensors, 25 samples per second
- E.g. jogging



(a) DyCA (b) ICA (c) PCA

DyCA – Application: C ECG data (from (2))

- 12 lead ECG, 500 samples per second
- One DyCA eigenvalue close to 1



(a) DyCA (b) ICA (c) PCA

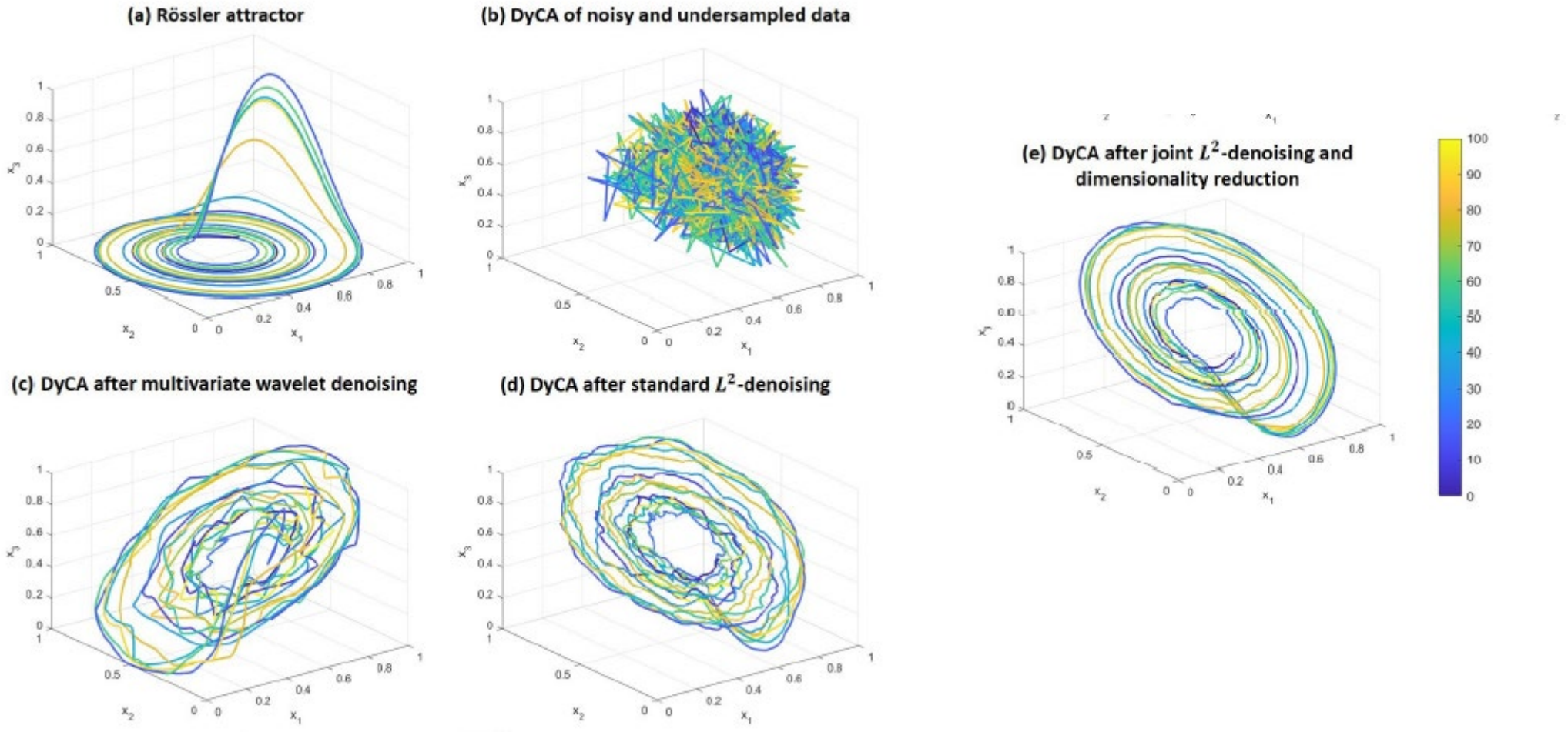
Extension: Robust DyCA

- in the case of high full rank additional noise
- joint variational signal denoising and dimensionality reduction approach
- minimization of

$$\underset{q,u_i,v_i\in\mathbb{R}^N}{\arg\min}\ \frac{1}{T}\sum_{j=1}^{T}\left[\frac{1}{2}\|q(t_j)-s(t_j)\|_2^2+\frac{\beta}{2}\|\partial_t q(t_j)\|_2^2\right.$$

$$\left.+\frac{\gamma}{2}\sum_{i=1}^{m}\|\partial_t q(t_j)^\top u_i-q(t_j)^\top v_i\|_2^2\right]$$

Robust DyCA: application example (from (3))



Conclusion: DyCA …

- is a method to decompose a multivariate signal into time-dependent amplitudes and corresponding multivariate modes
- is useful for multivariate signals obeying a specific set of coupled ODEs
- can remove noise components and detect the relevant subspace
- can be applied in many fields
  - to obtain low-dimensional trajectories in phase space (dimension reduction)
  - yielding possibly a feature for classification tasks
  - modeling a signal by data driven approach

DyCA - Dynamical Component Analysis: More Details

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(2) C. Uhl, M. Kern, M. Warmuth and B. Seifert, "Subspace Detection and Blind Source Separation of Multivariate Signals by Dynamical Component Analysis (DyCA)," in IEEE Open Journal of Signal Processing, vol. 1, pp. 230-241, 2020, doi: 10.1109/OJSP.2020.3038369

(3) M. Warmuth, P. Romberger, C. Uhl, Robust Dynamical Component Analysis via Multivariate Variational Denoising, Signal Processing Conference (EUSIPCO) 2021 29th European, pp. 2000-2004, 2021, doi: 10.23919/EUSIPCO54536.2021.9616108.

(4) P. Romberger, M. Warmuth, C. Uhl and K. Hüper, "Dynamical Component Analysis: Matrix Case and Differential Geometric Point of View", to be published in L. Brito Palma et al. (Eds.): CONTROLO 2022, LNEE 930, 2022, Springer Nature Switzerland

(5) C. Paglia, A. Stiehl, and C. Uhl, "Identification of Low-Dimensional Nonlinear Dynamics from High-Dimensional Simulated and Real-World Data", to be published in L. Brito Palma et al. (Eds.): CONTROLO 2022, LNEE 930, 2022, Springer Nature Switzerland